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Area preserving transformations in non-commutative space and NCCS theory

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Abstract. We propose a heuristic rule for the area transformation on the non-commutative plane. The non-commutative area preserving transformations are quantum deformations of the classical symplectic diffeomorphisms. The area preservation condition is formulated as a field equation in the non-commutative Chern–Simons gauge theory. A higher-dimensional generalization is suggested and the corresponding algebraic structure – the infinite-dimensional sin-Lie algebra – is extracted. As an illustrative example the second-quantized formulation for electrons in the lowest Landau level is considered.

1 Introduction

In the recent papers of [1–3] there is raised the intriguing question of the connection between hydrodynamics of the incompressible fluid and gauge field theory in non-commutative (NC) space. The practical realization of this idea for the planar ($D = 2$) electron system was considered earlier in the context of the Chern–Simons (CS) description of the quantum Hall effect [4, 5].

The introduction of the vector potential as a hydrodynamical variable together with the requirement of the invariance under the classical area preserving transformations leads to the CS gauge theory based on the group of the symplectic transformations Sdiff in $\mathbf{R}^2$. Non-commutative Chern–Simons (NCCS) theory is obtained by subjecting the classical symplectic structure to a quantum deformation.

In the present paper we propose to attribute the above deformation of the classical algebra to the non-commutativity of the two-dimensional surface under consideration. In other terms we consider a counterpart of area preserving diffeomorphisms (APDs) in the NC space and extract the corresponding symplectic structure, which, as one may expect, turns out to be a Moyal-type deformation of the classical Poisson bracket.

The non-commutative plane is represented by the pair of Hermitian operators $\hat{x}_i$ obeying $(i, k = 1, 2)$

$$[\hat{x}_i, \hat{x}_k] = \mathrm{i}\theta_{ik} = \mathrm{i}\theta\epsilon_{ik} \tag{1}$$

with the constant antisymmetric non-commutativity matrix $\boldsymbol{\theta}$ (for a review of NC geometry and adopted notation see e.g. [6]).

In order to establish the non-commutative analogue of APDs let us recall some basic definitions concerning classical symplectic structures and APDs [7, 8].

Let $\Delta \subset \mathbf{R}^2$ be some compact domain, described by the Cartesian coordinates x_i. The Poisson bracket is defined by

$$\{f(x), g(x)\}_{\mathrm{P}} = \theta_{ik}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_k}. \tag{2}$$

Consider a diffeomorphism

$$x_i \to x'_i = F_i(x), \qquad \Delta \to \Delta'. \tag{3}$$

Under this map the area ($\boldsymbol{\omega} = \boldsymbol{\theta}^{-1}$)

$$\Omega_\Delta = \int_\Delta \mathrm{d}^2x = \int_\Delta \mathrm{d}^2x \ \mathrm{Pf}\,\boldsymbol{\omega}\,\mathrm{Pf}\{x_i, x_k\}_{\mathrm{P}} \tag{4}$$

changes according to the rule

$$\Omega_\Delta \to \Omega'_\Delta = \int_{\Delta'} \mathrm{d}^2x' = \int_\Delta \mathrm{d}^2x \ \mathrm{Pf}\,\boldsymbol{\omega}\,\mathrm{Pf}\{F_i, F_k\}_{\mathrm{P}}. \tag{5}$$

Here we use that

$$\mathcal{J}(x) = \mathrm{Pf}\,\boldsymbol{\omega}\,\mathrm{Pf}\{F_i, F_k\}_{\mathrm{P}} \tag{6}$$

is the Jacobian determinant corresponding to the transformation (3). The Pfaffian is defined by $\mathrm{Pf}\,M_{ik} = (\det M_{ik})^{\frac{1}{2}}$.

Infinitesimal transformations

$$F_i(x) = x_i + \xi_i(x) \tag{7}$$

are generated by the divergenceless vector fields ξ_i,

$$\xi_i = \theta_{ik}\partial_k\xi, \qquad \partial_i\xi_i = 0. \tag{8}$$

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The algebraic structure sought can be revealed considering the variation (a Lie derivative) of the scalar function

$$\delta_{\boldsymbol{\xi}} f(x) = -\mathcal{L}_{\boldsymbol{\xi}} f(x) = \{\xi, f\}_{\mathrm{P}}. \tag{9}$$

The generators

$$t[\xi] = -\mathrm{i}\mathcal{L}_{\boldsymbol{\xi}} \tag{10}$$

satisfy the commutation relations

$$\Big[t[\xi], t[\eta]\Big] = t[\mathrm{i}\{\xi, \eta\}_{\mathrm{P}}], \tag{11}$$

which define the Lie algebra of the group Sdiff.

In the case of even-dimensional ($D = 2N$) Euclidean space one may assume that x_i are canonical coordinates, i.e. the only non-vanishing Poisson brackets are

$$\{x_{2\alpha-1}, x_{2\alpha}\} = \theta_{2\alpha-1,2\alpha} \equiv \theta_\alpha > 0, \tag{12}$$

where $\alpha = 1, 2, \ldots, N$.

In general the canonical coordinate system is not an orthonormal one, and the constant metric tensor h_{ik} is not diagonal. In that case under the diffeomorphism $x_i \to F_i(x)$ the D-volume changes according to the formula

$$\begin{aligned} \Omega_\Delta &= \int_\Delta \mathrm{d}^D x \sqrt{\det h_{ik}}\, \mathrm{Pf}\, \boldsymbol{\omega}\, \mathrm{Pf}\{x_i, x_k\}_{\mathrm{P}} \\ &\downarrow \\ \Omega'_\Delta &= \int_\Delta \mathrm{d}^D x \sqrt{\det h_{ik}}\, \mathrm{Pf}\, \boldsymbol{\omega}\, \mathrm{Pf}\{F_i, F_k\}_{\mathrm{P}}. \end{aligned} \tag{13}$$

For $D > 2$ divergenceless vector fields, (8) constitutes a symplectic (i.e. $\boldsymbol{\theta}$ conserving) subgroup of the volume preserving transformations.

2 Area preserving transformations in NC R^2

The formula (5) can be used in order to state the area transformation rule on the NC plane. But first one has to give mathematical substance to the notion of the area in the NC space.

For this purpose consider the realization of the commutation relation (1) in the Hilbert space $\mathcal{H}$. The operators $\hat{z} = \hat{x}_1 + \mathrm{i}\hat{x}_2$ and $\hat{\bar{z}} = \hat{x}_1 - \mathrm{i}\hat{x}_2$ satisfy the oscillator algebra

$$[\hat{z}, \hat{\bar{z}}] = 2\theta. \tag{14}$$

Introduce the normalized coherent states

$$|\zeta\rangle = \mathrm{e}^{-\frac{1}{4\theta}|\zeta|^2} \mathrm{e}^{\frac{1}{2\theta}\zeta\hat{\bar{z}}}|0\rangle, \tag{15}$$

with $\hat{z}|0\rangle = 0$ and $\langle\zeta|\zeta\rangle = 1$, such that

$$\hat{z}|\zeta\rangle = \zeta|\zeta\rangle \qquad \zeta = \zeta_1 + \mathrm{i}\zeta_2. \tag{16}$$

Here by ζ_i we denote the averages

$$\zeta_i = \langle\zeta|\hat{x}_i|\zeta\rangle. \tag{17}$$

Note that (17) establishes the one-to-one correspondence between the coherent states (15) and the points in $\mathbf{R}^2$ [9, 10]

$$(\zeta_1, \zeta_2) \in \mathbf{R}^2 \leftrightarrow |\zeta\rangle \in \mathcal{H}. \tag{18}$$

Using the isomorphism between the domain $\Delta \subset \mathbf{R}^2$ and the subspace $\mathcal{H}_\Delta \subset \mathcal{H}$, the area Ω_Δ can be represented as an integral in the ζ-plane,

$$\bar{\Omega}_\Delta = \int_\Delta \mathrm{d}^2\zeta\, \mathrm{Pf}\, \boldsymbol{\omega} \langle\zeta|\hat{\Omega}|\zeta\rangle \equiv \int_\Delta \mathrm{d}^2\zeta = \Omega_\Delta, \tag{19}$$

where

$$\hat{\Omega} = -\frac{\mathrm{i}}{2}\epsilon_{ij}[\hat{x}_i, \hat{x}_j]. \tag{20}$$

The matrix element in (19) mimics the Pfaffian $\mathrm{Pf}\{x_i, x_k\}_{\mathrm{P}}$ in (4). Acting by analogy with (5) we set the operator homomorphism

$$\hat{\mathcal{W}}[x_i] = \hat{x}_i \to \hat{\mathcal{W}}[F_i] \tag{21}$$

to induce the "area transformation"

$$\bar{\Omega}_\Delta \to \bar{\Omega}'_\Delta = \int_\Delta \mathrm{d}^2\zeta\, \mathrm{Pf}\, \boldsymbol{\omega} \langle\zeta|\hat{\Omega}'|\zeta\rangle, \tag{22}$$

where

$$\hat{\Omega}' = -\frac{\mathrm{i}}{2}\epsilon_{ik}\big[\hat{\mathcal{W}}[F_i], \hat{\mathcal{W}}[F_k]\big] = -\frac{\mathrm{i}}{2}\epsilon_{ik}\hat{\mathcal{W}}[F_i \star F_k]. \tag{23}$$

Here, by

$$\hat{\mathcal{W}}[F_i] = \frac{1}{(2\pi)^2}\int \mathrm{d}^2 p \int \mathrm{d}^2 x \mathrm{e}^{-\mathrm{i}p_i(\hat{x}_i - x_i)} F_i(x) \tag{24}$$

we denote the symbol of the Weyl ordering and

$$f(x) \star g(x) = \mathrm{e}^{\frac{\mathrm{i}}{2}\theta_{ik}\partial_i\partial'_k} f(x) \cdot g(x')|_{x'=x} \tag{25}$$

is the Groenewold–Moyal star product. It must be noted that the area transformation Ansatz (22) is not unique since one can use other types of operator ordering.

Note now that in the formula (22) there figures an integral

$$I_\Delta[f] \equiv \int_\Delta \mathrm{d}^2\zeta \langle\zeta|\hat{\mathcal{W}}[f]|\zeta\rangle \tag{26}$$

and from the definitions of coherent states and Weyl symbols one easily finds that

$$I_\Delta[f] = \int_\Delta \mathrm{d}^2\zeta \int \mathrm{d}^2 x D(\zeta - x) f(x), \tag{27}$$

where

$$D(\zeta - x) = \frac{1}{\pi\theta}\mathrm{e}^{-\frac{1}{\theta}(x_i - \zeta_i)^2}. \tag{28}$$

Thus the area transformation rule (22) takes the following form:

$$\bar{\Omega}_\Delta \to \bar{\Omega}'_\Delta \tag{29}$$

$$\bar{\Omega}'_\Delta = \int_\Delta \mathrm{d}^2\zeta \int \mathrm{d}^2 x D(\zeta - x)\,\mathrm{Pf}\,\boldsymbol{\omega}(-\mathrm{i})\,\mathrm{Pf}\{F_i, F_k\}_M(x),$$

where

$$\{F_i, F_k\}_M(x) = F_i(x) \star F_k(x) - F_k(x) \star F_i(x) \tag{30}$$

is the Moyal bracket.

Now we see that the transition to the NC case is realized by the substitution

$$\{F_i, F_k\}_\mathrm{P}(\zeta) \to \{F_i, F_k\}_\mathrm{NC}(\zeta) \tag{31}$$

$$\{F_i, F_k\}_\mathrm{NC}(\zeta) \equiv -\mathrm{i} \int \mathrm{d}^2 x D(\zeta - x)\{F_i, F_k\}_M(x).$$

In the commutative limit

$$\lim_{\theta\to 0} \mathrm{Pf}\,\boldsymbol{\omega}\,\mathrm{Pf}\{F_i, F_k\}_\mathrm{NC} = \lim_{\theta\to 0} \mathrm{Pf}\,\boldsymbol{\omega}\{F_i, F_k\}_\mathrm{P} = \mathcal{J} \tag{32}$$

is a Jacobian and the expression (29) gives the classical result (5). Note that the earlier proposed heuristic rule for the area transformation [11] may be represented in the form (29) if one sets $D(\zeta - x) = \delta^2(\zeta - x)$.

The requirement that (21) is an area preserving operator transformation results in

$$\Big[\hat{\mathcal{W}}[F_1], \hat{\mathcal{W}}[F_2]\Big] = \Big[\hat{x}_1, \hat{x}_2\Big] = \mathrm{i}\theta. \tag{33}$$

The same condition may be rewritten in terms of the Moyal bracket

$$-\mathrm{i}\,\mathrm{Pf}\,\boldsymbol{\omega}\,\mathrm{Pf}\{F_i, F_k\}_M = 1. \tag{34}$$

Equations (33) and (34) are non-commutative counterparts of the classical area preservation condition $\{F_1, F_2\}_\mathrm{P} = \theta$ and the corresponding operator homomorphisms can be referred to as the NC or quantum APDs.

The last equations can be generalized for the $2N$-dimensional non-commutative space defined by the commutators

$$\big[\hat{x}_{2\alpha-1}, \hat{x}_{2\alpha}\big] = \mathrm{i}\theta_{2\alpha-1,2\alpha} \equiv \mathrm{i}\theta_\alpha, \tag{35}$$

with $\alpha = 1, 2, \ldots, N$.

The Fock space operators are identified by

$$\hat{z}_\alpha = \hat{x}_{2\alpha-1} + \mathrm{i}\hat{x}_{2\alpha}, \qquad \hat{\bar{z}}_\alpha = \hat{x}_{2\alpha-1} - \mathrm{i}\hat{x}_{2\alpha} \tag{36}$$

and satisfy $\big[\hat{z}_\alpha, \hat{\bar{z}}_a\big] = 2\theta_\alpha$. Define coherent states

$$|\zeta_\alpha\rangle = \mathrm{e}^{-\frac{1}{4\theta_\alpha}|\zeta_\alpha|^2} e^{\frac{1}{2\theta_\alpha}\zeta_\alpha \hat{\bar{z}}_\alpha}|0\rangle, \tag{37}$$

with $\hat{z}_\alpha|0\rangle = 0$ and $\langle\zeta_\alpha|\zeta_\alpha\rangle = 1$, such that

$$\hat{z}_\alpha|\zeta_\alpha\rangle = \zeta_\alpha|\zeta_\alpha\rangle. \tag{38}$$

Construct the state vector

$$|\{\zeta\}\rangle = \prod_\alpha \otimes |\zeta_\alpha\rangle \tag{39}$$

and introduce averages

$$\zeta_i = \langle\{\zeta\}|\hat{x}_i|\{\zeta\}\rangle, \qquad 1 \le i \le D. \tag{40}$$

The last relation states the isomorphism between points in $\mathbf{R}^D$ and vectors in the Hilbert space.

The volume transformation can be represented as

$$\bar{\Omega}_\Delta \to \bar{\Omega}'_\Delta = \int_\Delta \mathrm{d}^D\zeta \sqrt{\det h_{ik}}\,\mathrm{Pf}\,\boldsymbol{\omega}\langle\{\zeta\}|\hat{\Omega}'|\{\zeta\}\rangle, \tag{41}$$

where

$$\hat{\Omega}' = \hat{\mathcal{W}}\big[\,\mathrm{Pf}_M\{F_i, F_k\}_M\big]. \tag{42}$$

The star-product modified Pfaffian is defined by

$$\mathrm{Pf}_M\,\mathbf{A} = \frac{\epsilon_{i_1 j_1 \ldots i_N j_N}}{2^N N!} A_{i_1 j_1} \star A_{i_1 j_1} \star \ldots \star A_{i_N j_N}, \tag{43}$$

where $\star$ and $\{F_i, F_j\}_M$ correspond to the matrix $\boldsymbol{\theta}$ in (35).

The multi-dimensional analogue of the volume preservation condition (34) may be represented as follows:

$$(-\mathrm{i})^N\,\mathrm{Pf}\,\boldsymbol{\omega}\,\mathrm{Pf}_M\{F_i, F_k\}_M = 1. \tag{44}$$

3 NCCS

Consider the map $(i, k = 1, 2, \ldots, 2N)$

$$x_i \to F_i(x) = x_i + \theta_{ik} a_k(x). \tag{45}$$

In the context of fluid mechanics this transformation might be viewed as a transition from the comoving (Lagrange) description to the Euler variables (covariant coordinates) and was considered in [2–4, 12].

The basic Moyal bracket is given by

$$\{F_i, F_k\}_M = \mathrm{i}\theta_{ik} + \mathrm{i}\Phi_{ik}, \tag{46}$$

where

$$\begin{aligned}\Phi_{ik} &= \theta_{im}\theta_{kn}[\partial_m a_n - \partial_n a_m - \mathrm{i}(a_m \star a_n - a_n \star a_m)] \\ &\equiv \theta_{im}\theta_{kn} f_{mn}.\end{aligned} \tag{47}$$

In terms of these hydrodynamical variables the volume preservation condition (44) looks as follows:

$$\sum_{l=0}^{N-1} \frac{N!}{l!(N-l)!}\Big[\epsilon_{i_1 j_1 \ldots i_N j_N} \times \theta_{i_1 j_1} \ldots \theta_{i_l j_l} \Phi_{i_{l+1}, j_{l+1}} \star \ldots \star \Phi_{i_N, j_N}\Big] = 0. \tag{48}$$

In $D = 2$

$$\mathrm{Pf}[\theta_{ik} + \Phi_{ik}] = \theta + \frac{1}{2}\theta^2 \epsilon_{mn} F_{mn}, \tag{49}$$

and (48) takes the form

$$\theta_{ik} D_i a_k \equiv Da = \theta_{ik}(\partial_i a_k - \mathrm{i} a_i \star a_k) = 0. \tag{50}$$

This equation is invariant uner the infinitesimal operator transformation

$$\begin{aligned} \Delta_\lambda \hat{\mathcal{W}}[F_i] &= -\mathrm{i}\Big[\hat{\mathcal{W}}[\lambda], \hat{\mathcal{W}}[F_i]\Big] \\ &= -\theta_{ik}\hat{\mathcal{W}}[\partial_k \lambda + \mathrm{i}\{\lambda, a_k\}_M] \\ &= -\theta_{ik}\hat{\mathcal{W}}[\delta_{\mathrm{gauge}} a_k] \end{aligned} \tag{51}$$

One can notice the evident resemblance between (50) and the Gauss law in the NCCS gauge theory. This theory is described by the Lagrangian ($\mu, \nu, \lambda = 0, 1, 2$)

$$\mathcal{L}_{\mathrm{NCCS}} = \frac{\kappa}{2}\varepsilon^{\mu\nu\lambda} a_\mu \star \left(\partial_\nu a_\lambda - \frac{\mathrm{i}}{3}\{a_\nu, a_\lambda\}_M\right) \tag{52}$$

and (50) turns out to be the Euler–Lagrange equation

$$\frac{\delta \mathcal{L}_{\mathrm{NCCS}}}{\delta a_0} = \kappa D a = 0. \tag{53}$$

Equation (51) is the gauge transformation for the vector potential.

In the commutative limit one recovers the CS theory with the classical gauge group Sdiff. The corresponding non-linear symplectic CS (SCS) Lagrangian is given by [13]

$$\mathcal{L}_{\mathrm{SCS}} = \frac{\nu}{2}\varepsilon^{\mu\nu\lambda} A_\mu \left(\partial_\nu A_\lambda + \frac{1}{3}\{A_\nu, A_\lambda\}_{\mathrm{P}}\right), \tag{54}$$

and the gauge transformations look as follows:

$$\delta_{\mathrm{gauge}} A_i = \partial_i \lambda - \{\lambda, A_i\}_{\mathrm{P}}. \tag{55}$$

Demanding gauge invariance under the group Sdiff one arrives at the Lagrangian (54) which could be interpreted as an approximation for the total NCCS Lagrangian (52). The transition from the SCS to the NCCS theory is accomplished by the replacement $\mathrm{i}\{f, g\}_{\mathrm{P}} \to \{f, g\}_M$. This scheme can be exploited with the goal to promote NCCS theory as an adequate scheme for the description of non-compressible quantum Hall fluids [4,5].

Note that the close link between non-commutative gauge fields and volume preserving diffeomorphisms is exposed in [2,3]. In these papers the area preservation condition is formulated as invariance of the basic commutator (1) under the map (45).

4 Algebraic structure and electrons in LLL

In this section we pass to the algebraic structure associated with the group of NC APDs and its explicit quantum-mechanical realization.

Consider the infinitesimal operator transformation

$$\hat{\mathcal{W}}[x_k] \to \hat{\mathcal{W}}[x_k + \theta_{kl}\partial_l \xi] = \hat{x}_k + \mathrm{i}\Big[\hat{\mathcal{W}}[\xi], \hat{x}_k\Big], \tag{56}$$

which is a non-commutative version of (8).

The corresponding variation of the Weyl symbol of the scalar function $f(x)$ will be given by

$$\Delta_\xi \hat{\mathcal{W}}[f] = -\mathrm{i}\Big[\hat{\mathcal{W}}[\xi], \hat{\mathcal{W}}[f]\Big] = \hat{\mathcal{W}}[-\mathrm{i}\{\xi, f\}_M]. \tag{57}$$

The generators

$$T[\xi] = \hat{\mathcal{W}}[\xi] \tag{58}$$

obey the commutation relation

$$\Big[T[\xi], T[\eta]\Big] = T[\{\xi, \eta\}_M] \tag{59}$$

in accord with (11).

The commutation relation (59) describes the algebraic structure of the group of symplectic diffeomorphisms in the non-commutative space. The corresponding structure constants could be fixed considering a special basis in the function space. In the case of operators

$$T_{\mathbf{p}} = T[\mathrm{e}^{\mathrm{i}\mathbf{p}\mathbf{x}}] \tag{60}$$

the commutation relation

$$\left[T_{\mathbf{p}}, T_{\mathbf{q}}\right] = -2\mathrm{i}\sin\left(\frac{1}{2}\theta_{ik} p_i q_k\right) T_{\mathbf{p}+\mathbf{q}} \tag{61}$$

reproduces the well-known algebra with a trigonometric structure constants [14]. Recall that originally the Lie brackets for the trigonometric sin-algebra were postulated by analogy with the Virasoro-type commutators and the corresponding structure constants were calculated imposing the Jacobi identities.

The application to the theory of the quantum Hall effect is based on the assumption that the configuration space of the system of electrons is a NC space. In the quantum Hall states the planar system of electrons is exposed to the intense orthogonal magnetic field $\mathbf{B} = (0, 0, -B)$ and electrons are constrained to lie in the lowest Landau level (LLL). Non-commutative coordinates satisfy

$$[\hat{r}_i, \hat{r}_k] = -\frac{\mathrm{i}}{B}\epsilon_{ik} \tag{62}$$

(we use natural units $c = \hbar = 1$ taking the electron charge $e = -1$). Recall that the commutator (62) arises from the Dirac bracket for the system with second class constraints [15,16].

For the one-particle quantum-mechanical density operator we take the Weyl symbol

$$\begin{aligned} \hat{\rho}_{\mathrm{QM}}(\mathbf{x}) &= \hat{\mathcal{W}}_{\mathbf{r}}[\delta(\mathbf{x} - \mathbf{r})] \\ &= \frac{1}{(2\pi)^2}\int \mathrm{d}\mathbf{k}\, \mathrm{e}^{-\mathrm{i}\mathbf{k}(\hat{\mathbf{r}} - \mathbf{x})}. \end{aligned} \tag{63}$$

The subscript $\mathbf{r}$ in (63) means that the Weyl ordering is taken with respect to the operators $\hat{r}_i$ satisfying (62). At the same time the coordinates x_i are considered as classical variables parameterizing the plane.

Note that

$$\int \mathrm{d}^2x\hat{\rho}_{\mathrm{QM}}(\mathbf{x}) = \hat{\mathcal{W}}_{\mathbf{r}}[1] \tag{64}$$

and

$$\int \mathrm{d}^2x\hat{\rho}_{\mathrm{QM}}\left[x_i+\theta_{ik}a_k(x)\right]$$
$$=\hat{\mathcal{W}}_{\mathbf{r}}\left[1-\frac{1}{2}\theta_{ij}f_{ij}+\frac{1}{2}\theta_{ij}\theta_{mn}(a_n\partial_m f_{ij}-f_{mj}f_{ni})\right]$$
$$+\mathcal{O}(\theta^3), \tag{65}$$

in accord with the Seiberg–Witten map [17].

The operator (63) obeys the commutation relation

$$\left[\hat{\rho}_{\mathrm{QM}}(\mathbf{x}'),\hat{\rho}_{\mathrm{QM}}(\mathbf{x}'')\right] = \int \mathrm{d}^2xK(\mathbf{x}',\mathbf{x}''|\mathbf{x})\hat{\rho}_{\mathrm{QM}}(\mathbf{x}). \tag{66}$$

In the kernel

$$K(\mathbf{x}',\mathbf{x}''|\mathbf{x}) = \delta(\mathbf{x}-\mathbf{x}')\star\delta(\mathbf{x}-\mathbf{x}'')$$
$$-\delta(\mathbf{x}-\mathbf{x}'')\star\delta(\mathbf{x}-\mathbf{x}') \tag{67}$$

the star product is implied with respect to the variable $\mathbf{x}$ and the non-commutativity parameter $\theta=-1/B$.

The charge operators

$$\hat{Q}_{\mathrm{QM}}\{\xi\} = \int \mathrm{d}^2x\hat{\rho}_{\mathrm{QM}}(\mathbf{x})\xi(\mathbf{x}) = \hat{\mathcal{W}}_{\mathbf{r}}[\xi] \tag{68}$$

generate the algebra (59):

$$\left[\hat{Q}_{\mathrm{QM}}\{\xi\},\hat{Q}_{\mathrm{QM}}\{\eta\}\right] = \hat{Q}_{\mathrm{QM}}\{\{\xi,\eta\}_M\}. \tag{69}$$

So far our consideration was restricted to one-particle quantum mechanics and it would be instructive to develop the corresponding field theory setup.

Introduce the operators

$$\hat{b}=\sqrt{\frac{B}{2}}(\hat{r}_1-\mathrm{i}\hat{r}_2),\qquad \hat{b}^+=\sqrt{\frac{B}{2}}(\hat{r}_1+\mathrm{i}\hat{r}_2) \tag{70}$$

(with $\left[\hat{b},\hat{b}^+\right]=1$) and oscillator states

$$|n\rangle=\frac{1}{\sqrt{n!}}\hat{b}^{+n}|0\rangle,\qquad \hat{b}|0\rangle=0 \tag{71}$$

and coherent states

$$\langle z| = \langle 0|\mathrm{e}^{\sqrt{\frac{B}{2}}\hat{b}}\mathrm{e}^{-\frac{B}{4}|z|^2}. \tag{72}$$

The LLL second-quantized field is given by

$$\hat{\psi}(\mathbf{x}) = \sum_{n=0}^{\infty}\hat{f}_n u_n(\mathbf{x}). \tag{73}$$

Here by $\hat{f}_n$ we denote the Fermi operators satisfying

$$\left[\hat{f}_n,\hat{f}_m^+\right]_+ = \delta_{mn}, \tag{74}$$

and u_n are one-particle LLL wave functions

$$u_n(\mathbf{x}) = \langle z|n\rangle, \tag{75}$$

which obey the LLL condition

$$\left(\partial_{\bar{z}}+\frac{B}{4}z\right)u_n(\mathbf{x})=0 \tag{76}$$

(we adopt the notation $z=x_1+\mathrm{i}x_2, 2\partial_{\bar{z}}=\partial_1+\mathrm{i}\partial_2$).

Recalling that as a one-particle density operator we use the Weyl symbol (63) we define the corresponding second-quantized objects: the density

$$\hat{\rho}(\mathbf{x}) = \sum_{m,n}\langle m|\rho_{\mathrm{QM}}(\mathbf{x})|n\rangle\hat{f}_m^+\hat{f}_n \tag{77}$$
$$=\int \mathrm{d}\mathbf{x}'\int \mathrm{d}\mathbf{x}''\hat{\psi}^+(\mathbf{x}')\langle z'|\hat{\rho}_{\mathrm{QM}}(\mathbf{x})|z''\rangle\hat{\psi}(\mathbf{x}'')$$

and the charge

$$\hat{Q}\{\xi\} = \int \mathrm{d}\mathbf{x}'\xi(\mathbf{x}')\hat{\rho}(\mathbf{x}'). \tag{78}$$

One easily verifies that

$$\left[\hat{Q}\{\xi\},\hat{Q}\{\eta\}\right] = \hat{Q}\{\{\xi,\eta\}_M\}. \tag{79}$$

The operator transformation

$$\varDelta_\xi\hat{r}_k = -\mathrm{i}\left[\hat{\mathcal{W}}_{\mathbf{r}}[\xi],\hat{r}_k\right] \tag{80}$$

induces the transformation of the oscillator states (71)

$$|n\rangle \to \left(1-\mathrm{i}\hat{\mathcal{W}}_{\mathbf{r}}[\xi]\right)|n\rangle \tag{81}$$

and the corresponding variation of the matter field

$$\delta_\xi\hat{\psi}(\mathbf{x}) = -\mathrm{i}\sum_{n=0}^{\infty}\langle z|\hat{\mathcal{W}}_{\mathbf{r}}[\xi]|n\rangle\hat{f}_n. \tag{82}$$

One easily finds that

$$\left(\partial_{\bar{z}}+\frac{B}{4}z\right)\delta_\xi\hat{\psi}(\mathbf{x})=0, \tag{83}$$

i.e. the transformation (82) does not violate the LLL condition. Note that essentially the same transformation was used in [18] with the aim to establish the algebra satisfied by the LLL projected density operators $\hat{\rho}^{\mathrm{L}}(\mathbf{x})$. The Fourier components of these densities obey the commutation relation [18, 19]

$$\left[\hat{\rho}^{\mathrm{L}}_{\mathbf{p}},\hat{\rho}^{\mathrm{L}}_{\mathbf{q}}\right] = 2\mathrm{i}\sin\left(\frac{\mathbf{p}\wedge\mathbf{q}}{2B}\right)e^{\frac{1}{2B}\mathbf{p}\cdot\mathbf{q}}\hat{\rho}^{\mathrm{L}}_{\mathbf{p}+\mathbf{q}}. \tag{84}$$

Note that the rescaled operators (60)

$$\tilde{T}_{\mathbf{p}} = \mathrm{e}^{-\frac{\theta}{4}\mathbf{p}^2} T_{\mathbf{p}} \tag{85}$$

obey the same algebra as the operators $\hat{\rho}^{\mathrm{L}}_{\mathbf{p}}$. The different forms of commutation relations are related to the various possible ways of operator ordering and definitions of the corresponding symbols. In the present paper we use Weyl symbols with the symmetric ordering of the Fock operators. Equally well one can apply other types of orderings and symbols (e.g. Wick normal and antinormal orderings) accompanied by the appropriate modifications of the star product.

5 Summary

In the present paper we introduce the notion of the finite area on the NC plane and suggest a heuristic rule for its transformations under the operator homomorphisms. The algebraic structure corresponding to APDs on the NC plane coincides with the quantum deformation of algebra of the group of the classical symplectic diffeomorphisms.

Invariance under NC APDs is equivalent of the Gauss law in the NCCS theory. In other words, the area preserving transformations in the NC space are induced by gauge potentials satisfying field equations in the NCCS gauge theory. The related gauge group corresponds to geometric transformations in the NC space.

APDs constitute an invariance group for the incompressible fluids like strongly interacting electrons in Laughlin states. This symmetry is embodied in the infinite-dimensional algebra generated by the LLL projected density operators [18,20]. On the other hand the standard CS gauge theory seems to be an adequate model for the description of the quantum Hall effect (see e.g. [21]). In the present paper we argue that the CS and the infinite symmetry approaches can be unified in the framework of the NCCS theory, where the gauge symmetry has a geometric origin.

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